EQUIVALENCE OF CERTAIN TYPES OF RHEOLOGICAL EQUATIONS OF STATE OF FLOWING POLYMER MEDIA. II. ANALYSIS OF SPECIFIC MODELS

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It is shown that many present rheological models reduce to the form of the equations of a generalized Maxwellian liquid.

The conditions under which the rheological equations of the relaxational and integral type can be reduced to the form of the generalized Maxwellian liquid with a symmetrical $\mathbf{B}_\beta$ tensor were established in [1]. In this paper, it is shown that these conditions are satisfied for many rheological equations, formulated in the literature. We are not attempting to cover all of the papers concerned with a development of rheological equations, but we shall limit ourselves to a number of modern models, illustrating the different dependences of the quantities $\varkappa_\alpha, \varphi_\alpha, \mathbf{B}_\beta$ in (17') and (17") [1] on the deformation parameters. The physical hypotheses, adopted in constructing the models, will be reflected in the nature of the expressions for $\varkappa_\alpha, \varphi_\alpha, \mathbf{B}_\beta$. Many rheological equations are constructed by generalizing the one-dimensional mechanical models, consisting of "springs" and "pistons" [2-5], to a three-dimensional form, invariant relative to the coordinate system used. The rheological equations, based on the mechanical Maxwell models, in general form are written as follows [2]:

$$\mathbf{T}' = \sum_\alpha \mathbf{T}_\alpha', \ \ \mathbf{T}_\alpha' + \lambda_\alpha \mathbf{F}_{abc}\mathbf{T}_\alpha' = 2\eta_\alpha \mathbf{D}, \tag{1}$$

where the linear operator F_{abc} [2], constructed by starting from the eight-constant Oldroid model, is the most general derivative of the symmetrical tensor $F_{abc}\mathbf{T} = \overset{\circ}{\mathbf{T}} + a(\mathbf{TD} + \mathbf{DT}) + b\mathbf{E}\,\mathrm{tr}\,\mathbf{TD} + c\mathbf{D}\,\mathrm{tr}\,\mathbf{T}$. Substituting into Eq. (1) $\mathbf{T}_\alpha' = \mathbf{T}_\alpha + (b/2a)\mathbf{E}\,\mathrm{tr}\,\mathbf{T}_\alpha$ and taking into account the fact that the stress tensor is determined to within an isotropic quantity, we arrive at the equivalent rheological model $\mathbf{T} = \sum_\alpha \mathbf{T}_\alpha$, where $\mathbf{T}_\alpha$ is determined from equations (17'), (17") [1] for:

$$\beta = 1,\ \mathbf{B}_1 = a\mathbf{D},\ \varkappa_\alpha = 1/\lambda_\alpha,\ \varphi_\alpha = -\left[\frac{\eta_\alpha}{a\lambda_\alpha^2} + \left(b + c + \frac{3cb}{2a}\right)\mathrm{tr}\,\mathbf{T}_\alpha\mathbf{D}\right],\ p_\alpha = \frac{1}{a}\left[\frac{\eta_\alpha}{\lambda_\alpha} - \left(\frac{b+c}{2} + \frac{3cb}{4a}\right)\mathrm{tr}\,\mathbf{T}_\alpha\right]. \tag{2}$$

In deriving relations (2), we used the expression $\frac{D\,\mathrm{tr}\,\mathbf{T}_\alpha}{Dt} + \frac{1}{\lambda_\alpha}\mathrm{tr}\,\mathbf{T}_\alpha = -2a\,\mathrm{tr}\,\mathbf{T}_\alpha\mathbf{D}$ obtained from (1). As is evident from (2), the parameters b and c enter only in the form of the combination $b + c + 3bc/2a$. The particular cases of Eq. (1) are: a Maxwellian liquid with a spectrum of relaxation times from the upper ($a = -1$, $b = c = 0$, $\omega_1 = \mathbf{F}_t^{-1\,T}(\tau)$) or lower ($a = 1$, $b = c = 0$, $\omega_1 = \mathbf{F}_t(\tau)$) convective derivatives, the Johnson–Segalman [6] model with nonaffine deformation (a is interpreted as the parameter for slipping of the lattice, $b = c = 0$), Spriggs' model [2-5] (α are integers, $\lambda_\alpha = \lambda/\alpha^\beta$, $\eta_\alpha = \eta/\alpha^\beta\varsigma(\beta)$, $\varsigma(\beta)$ is the Riemann zeta function, $a = -(1 + \varepsilon)$, $b = {}^2/_3(1 + \varepsilon)$, $c = 0$, the relation $b/a = -2/3$ is taken so that $\mathrm{tr}\,\mathbf{T}_\alpha' = 0$). The relation between the relaxational and integral equations is established for a Maxwellian liquid by Lodge [7], while for the Johnson–Segalman model, the relation is established in [6]. It is interesting to note that for (2) the tensor ω_β^T corresponds to the tensor introduced in [6]. For $a = b = c = 0$, Eq. (1) gives a Maxwellian liquid with a Jaumann derivative. Let us examine in greater detail the transition $a \to 0$ for $b = c = 0$ in the integral equation (18) from [1]. We shall seek the solution of Eq. (14) [1] in the form $\omega = \omega_0 - a\omega_0\mathbf{S} + O(a^2)$, $\omega_0|_{\tau=t} = \mathbf{E}$, $\mathbf{S}|_{\tau=t} = 0$. Substituting this expression into (14) [1] gives

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$$\frac{D\omega_0}{D\tau} = \mathbf{W}\omega_0,\ \frac{DS}{D\tau} = \omega_0^{\mathrm{T}}\mathbf{D}(\tau)\,\omega_0. \qquad (3)$$

From the second equation in (3) and the initial condition for it at $\tau = t$, it follows that the tensor $\mathbf{S}$ is symmetrical. As shown above (see (22), [1]), $\omega_0^{\mathrm{T}}\cdot\omega_0 = \mathbf{E}$. For this reason, $\lim_{a\to 0}\frac{1}{a}(\omega^{\mathrm{T}}\omega - \mathbf{E}) = -4\mathbf{S}(t, \tau)$. Thus, for a Maxwellian liquid with a Jaumann derivative the integral equation is written as follows: $\mathbf{T} = -\sum_\alpha \int_{-\infty}^{t} \frac{2\eta_\alpha}{\lambda_\alpha^2} \exp\left(-\frac{t-\tau}{\lambda_\alpha}\right)\mathbf{S}(t, \tau)\,d\tau$. Integrating this expression by parts, we arrive at the integral relation derived in [5, 8, 9]. The rheological equations, based on the mechanical Jeffries model, are written in general form [2-5]:

$$\mathbf{T}' = \sum_\alpha \mathbf{T}'_\alpha,\ \mathbf{T}'_\alpha + \lambda_{1\alpha}\mathbf{F}_{a_1b_1c_1}\mathbf{T}'_\alpha = 2\eta_\alpha(\mathbf{D} + \lambda_{2\alpha}\mathbf{F}_{a_2b_20}\mathbf{D}). \qquad (4)$$

The last term in the operator $\mathbf{F}_{abc}$ is absent on the right side of (4), since $\mathrm{tr}\,\mathbf{D} = 0$. For $a_1 = a_2 = a$, just as in (1), Eq. (4) is reduced to the equivalent model $\mathbf{T} = 2\left(\sum_\alpha \eta_\alpha \frac{\lambda_{2\alpha}}{\lambda_{1\alpha}}\right)\cdot \mathbf{D} + \sum_\alpha \mathbf{T}_\alpha$, where $\mathbf{T}_\alpha$ is determined from the system (17), (17") [1] for $\varkappa_\alpha = \frac{1}{\lambda_\alpha}$, $\varphi_\alpha = -\left[\frac{\eta_\alpha(1-\lambda_{2\alpha}\lambda_{1\alpha})}{a\lambda_{1\alpha}^2} + \left(b_1 + c_1 + \frac{3c_1b_1}{2a}\right)\mathrm{tr}\,\mathbf{T}_\alpha\mathbf{D} + \frac{2\eta_\alpha\lambda_{2\alpha}}{\lambda_{1\alpha}}\left(1 + \frac{3c_1}{2a}\right)(b_1 - b_2)\,\mathrm{tr}\,\mathbf{D}^2\right]$, $\beta = 1$, and $\mathbf{B}_1 = a\mathbf{D}$. The quantity $\sum_\alpha \eta_\alpha\lambda_{2\alpha}/\lambda_{1\alpha}$ is the limiting value of the shear viscosity for "fast" actions, for example, $\dot{\gamma} \to \infty$ in stationary and $\omega \to \infty$ in oscillating shear flows. For $a_1 \neq a_2$, Eq. (4) does not reduce to (17'), (17") [1], which is related to the breakdown of the necessary condition for the transition. The following models are particular cases of Eq. (4) with $a_1 = a_2$ [4, 5]: the three-constant Oldroid ($a = -1$, $b_1 = b_2 = 2/3$, $c_1 = 0$) and Williams ($a = -1$, $b_1 = b_2 = 2W$, $c_1 = 0$). For calculations of flows in which the main rheological factor is the dependence of the properties on the deformation, a modification of the equations of a Maxwellian liquid with an upper convective derivative and a single relaxation time, obtained by introducing empirical equations for the dependence of the viscosity and relaxation time on the deformation rate tensor $\lambda = F_1(\mathrm{II}_\mathbf{D})$, $\eta = F_2(\mathrm{II}_\mathbf{D})$ [3-5], where $\mathrm{II}_\mathbf{D} = -\frac{1}{2}\mathrm{tr}\,\mathbf{D}^2$, is used. This model corresponds to the differential (17') [1] and integral (19) [1] equations with $\varkappa = 1/F_1$, $p = -\tilde{\varphi} = -F_2/F_1$, $\mathbf{B} = -\mathbf{D}$. Sometimes the condition $G = \eta/\lambda = \mathrm{const}$ is added; then $F_2 = GF_1$. More complicated models are the integral models (18) and (19) in [1], in which $\varkappa_\alpha$, φ_α in (18) [1] and $\varkappa_\alpha$, $\tilde{\varphi}_\alpha$ in (19) [1] depend on the deformation rate tensor. In general form, they are written as follows: $\beta = 1, 2$, $\varkappa_{\beta\alpha} = 1/\lambda_{\beta\alpha}g_{\beta\alpha}(\mathrm{II}_\mathbf{D})$, $\varphi_{\beta\alpha} = \eta_{\beta\alpha}f_{\beta\alpha}(\mathrm{II}_\mathbf{D})/\lambda_{\beta\alpha}^2$, $\mathbf{B}_1 = -\mathbf{D}$, $\mathbf{B}_2 = \mathbf{D}$. In models of the type (19) [1], similar equations are used for $\varkappa_{\beta\alpha}$ and $\tilde{\varphi}_{\beta\alpha}$. For them, p_α depends only on the instantaneous value of $\mathrm{II}_\mathbf{D}$. In most cases, $\eta_{1\alpha} = \eta_{2\alpha}$, $\lambda_{1\alpha} = \lambda_{2\alpha}$, $g_{1\alpha} = g_{2\alpha}$, $f_{1\alpha} = \left(1 + \frac{\varepsilon}{2}\right)f_\alpha$, $f_{2\alpha} = -(\varepsilon/2)f_\alpha$. The quantity ε, which depends on $\mathrm{II}_\mathbf{D}$, is proportional to the ratio of the second and first differences of the normal stresses in a stationary shear flow. The models are distinguished only by the choice of ε and equations for f_α, g_α, η_α, λ_α. In a number of cases, the viscosity of the solvent is taken into account and $2\eta_s\mathbf{D}$ is added to the integral relation (19) [1]. In recent years, there has been an intense development of the relaxation models with characteristics depending on the stress tensor. Usually the dependences are taken as functions of the quantity $E_\alpha = (1/2)\mathrm{tr}\,\mathbf{T}_\alpha$, which is interpreted as the elastic energy of the Gaussian lattice. Such models require only several adjustable parameters in order to describe satisfactorily polymer media both in shear and in elongational flows. In [10], the following model is proposed: $\mathbf{T} = \Sigma\mathbf{T}_\alpha$, $\mathbf{T}_\alpha + \lambda_\alpha G_\alpha(\mathbf{T}_\alpha^{\nabla}/G_\alpha) = 2\lambda_\alpha G_\alpha\mathbf{D}$, $\lambda_\alpha\frac{Dx_\alpha}{Dt} = 1 - x_\alpha - ax_\alpha\sqrt{E_\alpha/G_\alpha}$, $G_\alpha = G_{0\alpha}x_\alpha$, $\lambda_\alpha = \lambda_{0\alpha}x_\alpha^n$, where a is an adjustable parameter; $n = 1.4$; (∇) is the upper convective derivative [2]. By introducing $p_\alpha = -G_{0\alpha}x_\alpha$, this model reduces to the system (17'), (17") [1] with $\beta = 1$, $\mathbf{B}_1 = -\mathbf{D}$, $\varphi_\alpha = -p_\alpha/\lambda_\alpha$, $\varkappa_\alpha = \frac{1}{2}(2 + G_{0\alpha}/p_\alpha + a\sqrt{-E_\alpha/p_\alpha})$, $\lambda_\alpha = \lambda_{0\alpha}(-p_\alpha/G_{0\alpha})^n$. In [11], an equation of the Maxwell liquid type with an upper convective derivative and a single relaxation time, constructed starting from an examination of the dynamics of a two-centered dumbbell, is modified by introducing the dependences λ and η as a function of the quantity (tr $\xi - 1$), where the tensor ξ characterizes the stretching of the macromolecules. In this case, the shear modulus G is assumed to be constant. The rheological equation [11] for the excess stress tensor $\mathbf{T} = 3\,G\xi$ reduces to Eq. (17') [1] with $\alpha = \beta = 1$, $\varkappa = 1/\lambda$, $p = -G$, $B_1 = -\mathbf{D}$, $\lambda = F(2\,E/3\,G)$, where the function F is determined from the experimental data. In [12], the equation for the Maxwellian liquid with the upper convective derivative is also modifed by taking into account the dependences of λ_α and η_α on E_α; G_α is assumed to be constant. This model is written in the form (17') [1] with $\varkappa_\alpha = 1/\lambda_\alpha$, $\beta = 1$, $\mathbf{B}_1 = -\mathbf{D}$, $p_\alpha = -G_\alpha$, $\lambda_\alpha = \lambda_{0\alpha}$ exp $(1/f_\alpha - 1/f_{0\alpha})$, $f_\alpha = f_{0\alpha} + aE_\alpha/G_\alpha$. Here, f_α is the volume fraction of the free volume; a is an adjustable parameter. In [13], this model was developed further. It is assumed that f_α relaxes: $\lambda_\alpha \dfrac{Df_\alpha}{Dt} = bH_\alpha \dfrac{E_\alpha}{G_\alpha} - (f_\alpha - f_0)$, $G_\alpha = H_\alpha \Delta_\alpha \ln \lambda_0$, $H = H(\lambda_{0\alpha})$, where $H(\lambda)$ is the spectral function, b is an adjustable parameter, and the discretization of the spectrum is carried out by separating it into intervals $\Delta_\alpha \lambda_0$. In [14], starting from the kinetic theory of a rearranging lattice developed in [15], a model is constructed in which both the dependence of the relaxation time λ_α on the stress tensor ($Y(\mathbf{T}_\alpha)$) and slipping (the parameter ξ depending on $II_\mathbf{D}$) of the lattice relative to the medium are taken into account. The shear modulus is assumed to be constant. This model corresponds to Eqs. (17'), (19) [1] with $\beta = 1$, $B_1 = -(1 - \xi)\mathbf{D}$, $\varkappa_\alpha = Y(\mathbf{T}_\alpha)/\lambda_\alpha$, $\tilde{\varphi}_\alpha = G_\alpha/(1-\xi)$. In this model, the tensor ω_β^T coincides with the tensor in the Johnson–Segalman model [6]. Two equations are proposed for the function Y: $Y = 1 + \varepsilon E_\alpha/G_\alpha$, $Y = \exp(\varepsilon E_\alpha/G_\alpha)$. The quantity ξ is found from the relation of the second and first differences of the normal stresses with a stationary shear flow, while ε is found from data on the viscosity with uniaxial stretching. The rheological models examined above correspond to the Kaye's type integral equation with parameters depending on the stress tensor. Since for an incompressible liquid the stress tensor is determined to within an isotropic term, it is proposed in [16] that the dependence of rheological parameters on Q_1 and Q_2, independent of the isotropic term, be given: $Q_1 = I_T^2 - 3II_T$, $Q_2 = 2I_T^3 - 9I_TII_T + 27III_T$, $I_T = \operatorname{tr}\mathbf{T}$, $II_T = \frac{1}{2}(\operatorname{tr}^2\mathbf{T} - \operatorname{tr}\mathbf{T}^2)$, $III_\mathbf{T} = \det \mathbf{T}$. Models proposed in [17, 18] also are of the same type. For them in (18) [1], $\beta = 1$, $B_1 = -\mathbf{D}$, $\varkappa_\alpha = 1/\lambda_\alpha g_\alpha$, $\varphi_\alpha = \eta_\alpha f_\alpha/\lambda_\alpha^2$. In [17], the functions g_α, f_α depend on the second invariant $II_{T\alpha}$. It is shown in [18] that the best agreement with experimental data is achieved when the dependence of g_α, f_α on E is introduced. Up to the present time, a large number of papers have been published in which a microscopic approach is used to study the dynamics of viscoelastic liquids. In these papers, various modifications of the Kargin–Slonimskii–Rauz model are used, in which the macromolecule is modelled as a collection of particles connected by springs and submerged in a Newtonian liquid. Although this approach is essentially phenomenological, it leads to many important results. Reviews of investigations along microscopic lines are given in [3, 4, 19]. As is well known, these models lead to equations of a Maxwellian liquid with an upper covariant derivative, a discrete spectrum of relaxation times, and constant coefficients in the equations. This rheological model does not give a correct description of a viscoelastic liquid with large deformations, which is essentially related to the assumption of linearity of the springs connecting the particles. Taking into account the unlimited growth in the rigidity of the springs with total stretching of the macromolecules, as follows from a molecular theory, eliminates this shortcoming. The nonlinearity of the interaction forces between particles greatly complicates the problem and this factor is predominantly analyzed only for two particles: dumbbells, which leads to a single relaxation time. In contrast to most papers investigating the behavior of a dilute solution of dumbbells in shear and elongational flows, the approximation of a slow flow, and so forth, rheological equations are obtained in [20-23] for such a medium. The approach in [20, 21] is based on a system of equations for different moments of the distribution function of dumbbells with respect to length and direction $\vec{b}$. Starting from the reasonable physical hypothesis that this distribution is close to a delta function, all moments are expressed in terms of the quantities $\langle b_i b_j \rangle$. As a result, a rheological equation is obtained that is valid for an arbitrary flow. Rheological models, obtained in [20-22], can be written as follows: $\mathbf{T}' = 2\eta_s\mathbf{D} + \mathbf{T}_p$; a, b, c, and d depend on $E_p = \frac{1}{2}\operatorname{tr}\mathbf{T}_p$; $(a\overset{0}{\mathbf{T}}_p) - b(\mathbf{D}\mathbf{T}_p + \mathbf{T}_p\mathbf{D}) + c\mathbf{T}_p = d\mathbf{E}$. By introducing $\mathbf{T} = \mathbf{T}_p + p\mathbf{E}$, these models reduce to the form (17'), (17") [1], with $\mathbf{B} = -\dfrac{b}{a}\mathbf{D}$,

$$\varphi = \frac{d}{a},\quad a' = \frac{1}{2}\frac{da}{dE_p},\quad \varkappa = \frac{1}{2}\left[\frac{2a'b\operatorname{tr}\mathbf{TD}}{a + a'(\operatorname{tr}\mathbf{T} - 3p)} + \frac{3a'd + ca}{a + a'(\operatorname{tr}\mathbf{T} - 3p)}\right].$$

For the models in [20, 21]

$\alpha = b$, $1/\alpha = K(1 + 2E_p/A)$; then $\varkappa = cK\left[1+\frac{\mathrm{tr}\,\mathbf{T}-3p}{A}\right]^2 - 3dK\frac{1}{A}\left(1+\frac{\mathrm{tr}\,\mathbf{T}-3p}{A}\right)-\frac{2}{A}\,\mathrm{tr}\,\mathbf{TD}$.

In [23], a rheological equation is derived by expanding the distribution function of dumbbells in a series with respect to the quantity ε, characterizing the small nonlinearity of the springs. In the first approximation in ε (λ, G, ε are the dumbbell parameters): $\mathbf{T}' = 2\eta_s\mathbf{D} + \mathbf{T}_p$, $\mathbf{T}_p = G[\boldsymbol{\alpha} + \varepsilon(\boldsymbol{\alpha}\,\mathrm{tr}\,\boldsymbol{\alpha} + 2\boldsymbol{\alpha}^2) - \mathbf{E}]$, $\boldsymbol{\alpha} + \lambda\overset{\nabla}{\boldsymbol{\alpha}} + \varepsilon(\boldsymbol{\alpha}\,\mathrm{tr}\,\boldsymbol{\alpha} + 2\boldsymbol{\alpha}^2) = \mathbf{E}$. Introducing $\mathbf{T} = G\left[\boldsymbol{\alpha} + \varepsilon(\boldsymbol{\alpha}\,\mathrm{tr}\,\boldsymbol{\alpha} + 2\boldsymbol{\alpha}^2) + \frac{\varepsilon}{2}(\mathrm{tr}^2\boldsymbol{\alpha} - \mathrm{tr}\,\boldsymbol{\alpha}^2)\mathbf{E}\right] + p\mathbf{E}$ and eliminating the tensor $\boldsymbol{\alpha}$, we arrive, up to terms of order ε^2, at the system (17'), (17") [1] with $\varkappa = \frac{1}{\lambda}\left[1 - 7\varepsilon + \frac{2\varepsilon}{G}(\mathrm{tr}\,\mathbf{T} - 5p)\right]$, $\varphi = \frac{1}{\lambda}\left\{1 + \frac{3\varepsilon}{G}\cdot(\mathrm{tr}\,\mathbf{T} - 3p) + \frac{\varepsilon}{G^2}\left[\frac{1}{2}(\mathrm{tr}\,\mathbf{T}^2 - \mathrm{tr}^2\,\mathbf{T}) - 3p^2 + 2p\,\mathrm{tr}\,\mathbf{T} + 4(\mathrm{tr}\,\mathbf{T}\,\mathrm{tr}\,\mathbf{TD} - \mathrm{tr}\,\mathbf{T}^2\mathbf{D} - p\,\mathrm{tr}\,\mathbf{TD})\right]\right\}$, $\mathbf{B} = -\mathbf{D} + \frac{3\varepsilon}{\lambda G}\mathbf{T} - \frac{2\varepsilon}{G}\cdot(\mathbf{TD} + \mathbf{DT} - \mathbf{D}\,\mathrm{tr}\,\mathbf{T} + p\mathbf{D})$.

As in the case of Zimm's model [3, 4, 19], rheological equations of a dilute solution of dumbbells with nonlinear springs coincide in form with the rheological equations derived from the lattice theory. However, this correspondence is formal. A unique approach to constructing a rheological model, based on nonequilibrium thermodynamics, is developed in [24]. This model includes an equation that expresses the elastic potential W in terms of the invariants of the deformation tensor **C**. For the Mooney–Rivlin potential $W = \mu[I_C - 3 + \alpha(II_C - 3)]$, the model in [24] with a single relaxation time is written as ($q\eta$ = const, det **C** = 1): $\mathbf{T}' = 2\eta\mathbf{D} + 2\mu\mathbf{C} - 2\mu\alpha\,\mathbf{C}^{-1}$, $\overset{\nabla}{\mathbf{C}} + 2\mathbf{C}\mathbf{e}_p = 0$, $\mathbf{e}_p = q\left[\mathbf{C} - \mathbf{C}^{-1} + \frac{1}{3}(II_C - I_C)\mathbf{E}\right]$, $q(\mathbf{C}) = q_0\exp\left[-\frac{1}{2}(1+\alpha)\beta(I_C + II_C - 6)\right]$. By introducing $\mathbf{T}_1 = 2\mu\mathbf{C} + p_1\mathbf{E}$ and $\mathbf{T}_2 = -2\alpha\mu\mathbf{C}^{-1} + p_2\mathbf{E}$, this model is reduced to Eqs. (17'), (17") [1] with $\mathbf{B}_1 = -\mathbf{D} + \frac{q_1}{2\mu}\mathbf{T}_1$, $\varkappa_1 = \frac{2}{3}q_1\varsigma_1 - \frac{q_1 p_1}{\mu}$, $\varphi_1 = 4q_1\mu$, $q_1 = q\left(\frac{\mathbf{T}_1 - p_1\mathbf{E}}{2\mu}\right)$, $\mathbf{B}_2 = \mathbf{D} - \frac{q_2}{2\alpha\mu}\mathbf{T}_2$, $\varkappa_2 = -\frac{2}{3}q_2\varsigma_2 + \frac{q_2 p_2}{\alpha\mu}$, $\varphi_2 = -4q_2\mu$, $q_2 = q\left(\frac{\mathbf{T}_2 - p_2\mathbf{E}}{2\alpha\mu}\right)$; $\varsigma_1 = II_{(\mathbf{T}_1 - p_1\mathbf{E})/2\mu} - I_{(\mathbf{T}_1 - p_1\mathbf{E})/2\mu}$, $\varsigma = -II_{(\mathbf{T}_2 - p_2\mathbf{E})/2\alpha\mu} - I_{(\mathbf{T}_2 - p_2\mathbf{E})/2\alpha\mu}$. Starting from these rheological equations with the conditions $p_1/2\mu \to -1$, $p_2/2\mu\alpha \to 1$ for $t \to -\infty$, it can be shown that $\det\left(\frac{\mathbf{T}_1 - p_1\mathbf{E}}{2\mu}\right) = 1$, $\left(\frac{\mathbf{T}_2 - p_2\mathbf{E}}{2\alpha\mu}\right)\cdot\left(\frac{\mathbf{T}_1 - p_1\mathbf{E}}{2\mu}\right) = -\mathbf{E}$. The model in [24] with a spectrum of relaxation times reduces to the form (17'), (17") [1] by transformations similar to the case involving a single relaxation time. The analysis carried out above shows that many rheological models can be represented in the form of equations of a generalized Maxwellian liquid: differential (17'), (17") [1] or equivalent integral (18), (19) [1]. In addition, only the first several terms of the general equations (see [1]) are used for the tensor **B**, expressing it in terms of the deformation rate and stress tensors. In equations for the scalars $\varkappa$ and φ, the complete system of invariants of these tensors is also not used. In the case of a spectrum of relaxation times, each relaxation oscillator is described by its own equation, i.e., there is no interaction between relaxation oscillators (see (3), (4), (9) [1]). However, in the literature, there are a number of integral models which cannot be represented in the form of equations of the generalized Maxwellian liquid. This is related to two factors: giving in (18), (19) [1] the dependences of the quantities $\varkappa$, φ on the time t, and using, as a measure of the deformation, tensors that do not have the multiplicative property (16) [1]. The first factor occurs in the models in [3-5], where the tensors $\omega_1(t, \tau) = \mathbf{F}_t(\tau)$ and $\omega_2(t, \tau) = \mathbf{F}_t^{-1\mathrm{T}}(\tau)$ and the functions $\varkappa_\alpha(\tau)$, $\varphi_\alpha(t, \tau)$ are used. In the Bog–White equations [3-5], the quantity φ_α depends on the average value of the second invariant of the deformation rate tensor $\frac{1}{t-\tau}\int_\tau^t II_{\mathbf{D}}^{1/2}\,d\xi$ (a more complicated dependence on $\frac{1}{t-\tau}\int_\tau^t F(II_{\mathbf{D}})\,d\xi$ is also proposed), and for [25], on $|II_{\mathbf{D}}(t) - II_{\mathbf{D}}(\tau)|^{1/2}$. In the rheological Tanner–Simmons equation [2-5], $\varphi_\alpha(t, \tau) = \varphi_\alpha$ if $II_{C_t}(\tau) \leqslant K^2 + 3$, and $\varphi_\alpha(t, \tau) = 0$ if $II_{C_t}(\tau) > K^2 + 3$. The parameter K, determining the limiting deformation of the lattice, is interpreted as its strength. For the model in [3], $\varphi_\alpha(t, \tau) = \varphi_\alpha$ if $\varkappa_\alpha > S(t)$, and $\varphi_\alpha(t, \tau) = 0$ if $\varkappa_\alpha \leqslant S(t)$. The function S(t) depends on the past history of the deformation and is related to the critical energy of breakdown of the spectrum. In BKZ type models [2-5], to which the Tanner–Simmons equation is also related: $\varphi_\alpha = \varphi_{0\alpha} h(I_{C_t(\tau)}, II_{C_t(\tau)})$. In papers developing this line of thought, different equations are used for the function h. An even greater complication is proposed in [26]: $\varkappa_\alpha = f_{1\alpha}[II_D(t), II_D(\tau), II_D(\xi)]$, $\varphi_\alpha = \frac{1}{t-\tau}\int_{2\alpha}^{t} f_{2\alpha}[II_D(t), II_D(\tau), II_D(\xi)]\,d\xi$. In a number of papers (see [27]), it is proposed that φ_α = const and other tensors F be used as measures of deformation. However, in these papers, only tensors that depend nonlinearly on $C_t(\tau)$ are proposed. In view of the isotropy of such a function, it can be represented in the form $F = h_0 E + h_1 C_t(\tau) + h_2 C_t^{-1}(\tau)$, where h_0, h_1, h_2 depend on $I_{C_t}(\tau)$, $II_{C_t}(\tau)$. Thus, these models actually reduce to BKZ type models. The conditons under which the relaxational and integral equations are equivalent for an arbitrary flow were established in [1]. However, the rheological equations may turn out to be equivalent for certain types of flows. The possibility of such a situation must be foreseen in comparing the experimental data with calculations on rheological models of different types. Thus, for example, let us examine the motion with a commutative history [28]. For this motion, for each time t, there exists a configuration * such that $F_*(\tau) = Q(\tau)F(\tau)$ for all $\tau \leqslant t$, where $Q(\tau)$ is an orthogonal tensor, $F(\tau_1)F(\tau_2) = F(\tau_2)F(\tau_1)$. It is proved in [28] that then $F(\tau) = \exp[M(\tau)]$, where $M(\tau_1)M(\tau_2) = M(\tau_2)M(\tau_1)$. For such a flow, $C_t(\tau) = Q(t)\exp[M(\tau) + M^T(\tau) - M(t) - M^T(t)]Q^T(t)$, $D(t) = Q(t)M(t)Q^T(t)$, $W(t) = Q(t)Q^T(t)$. Using these relations, it is not difficult to show that for a commutative stretching flow, when $M(t) = \mathrm{diag}[\varepsilon_1(t), \varepsilon_2(t), \varepsilon_3(t)]$, the rheological equation of the BKZ type $\mathbf{T} = \sum_\alpha \mathbf{T}_\alpha$, $\mathbf{T}_\alpha = \int_{-\infty}^{t} \varphi_\alpha(\tau)\exp\left[-\int_\tau^t \varkappa_\alpha(\xi)\,d\xi\right](\mathbf{C}_t^\alpha(\tau) - \mathbf{E})\,d\tau$ is equivalent to (17') and (17") [1] with $B_\alpha = \alpha D$. (For simple stretching, the analogous result was obtained in [27].) In the case of an arbitrary flow, these relaxational and integral equations are equivalent only with $\alpha = \pm 1$. The kinematics of a commutative stretching flow is analyzed in [28]. The basic disadvantage of rheological models, not leading to equations of the generalized Maxwellian liquid, is the fact that in the numerical solution of problems of hydrodynamics and heat transfer, it is not possible to use the finite difference methods, traditionally used in fluid mechanics. Computational methods are required in which the motion of separate liquid particles is examined. This significant complication is related to the necessity of calculating the integrals along the trajectory of the fluid particles in calculating the stress tensor. The possibility of writing many rheological models in the unified form of the equations of the generalized Maxwellian liquid shows the promise of further analysis of these equations, in particular, taking into account a large number of terms in the expression for the tensor B and the functions $\varkappa$, φ [1]. A unified form for writing many rheological equations also permits developing unified programs for numerical solution for problems in fluid mechanics of a viscoelastic fluid. Comparing the results of calculations with experiment, it is possible to choose for a given fluid a rheological equation, which would describe its behavior with nonstationary flow and complex flow geometry, and not only in simple viscosymmetric situations. Carrying out numerical calculations of the same problems in rheodynamics and heat transfer using rheological equations of state of different types will also permit clarifying the importance of the difference between the hydrodynamic and thermal characteristics, predicted by different models. Up to the present time, in view of the necessity for developing computer programs, practically specially for each rheological equation of state, such investigations have not been carried out. The numerical calculations of hydrodynamic and heat problems, presented in the literature, were carried out for some specific rheological model, chosen in a fairly arbitrary manner.

LITERATURE CITED

1. B. M. Khusid, "Equivalence of some types of rheological equations of state of flowing polymer media. I. General analysis," Inzh.-Fiz. Zh., 42, No. 4, 670-677 (1982).
2. J. Astarita and J. Marucci, Foundations of the Hydromechanics of Non-Newtonian Liquids [Russian translation], Mir, Moscow (1978).
3. G. V. Vinogradov and A. Ya. Malkin, Rheology of Polymers [in Russian], Khimiya, Moscow (1977).
4. Chang Dei Khan, Rheology in Polymer Reprocessing Processes [Russian translation], Khimiya, Moscow (1979).

5. R. B. Bird, R. C. Armstrong, and O. Hassager, Dynamics of Polymeric Fluids, Vol. I, Wiley, New York (1977).
6. M. W. Johnson and D. Segalman, "A model for viscoelastic fluid behavior which allows nonaffine deformation," J. Non-Newtonian Fluid Mech., 2, No. 3, 255-270 (1977).
7. A. S. Lodge, Elastic Liquids [in Russian], Nauka, Moscow (1968).
8. J. D. Goddard and C. Miller, "An inverse for the Jaumann derivative and some applications to the rheology of viscoelastic fluids," Rheol. Acta, 5, No. 3, 177-184 (1966).
9. V. P. Gorodtsov and A. I. Leonov, "Kinematics, nonequilibrium thermodynamics, and rheological relations in the nonlinear theory of viscoelasticity," Prikl. Mat. Mekh., 32, No. 1, 70-94 (1968).
10. D. Acierno, F. P. La Mantia, G. Marrucci, and G. Titomanlio, "A nonlinear viscoelastic model with structure-dependent relaxation times. I. Basic formulation," J. Non-Newtonian Fluid Mech., 1, 125-146 (1976).
11. V. N. Pokrovskii and N. P. Kruchinin, "Nonlinear effects accompanying flow of linear polymers," Vysokomolekul. Soed. Kratkie Soobshch., 22, No. 5, 335-338 (1980).
12. F. P. La Mantia, "Nonlinear viscoelasticity of polymeric liquids interpreted by means of a stress-dependent free volume," Rheol. Acta, 16, No. 3, 302-308 (1977).
13. F. P. La Mantia and G. Titomanlio, "Testing of a constitutive equation with free volume dependent relaxation spectrum," Rheol. Acta, 18, No. 4, 469-477 (1979).
14. N. Phan-Thien, "A nonlinear network viscoelastic model," J. Rheology, 22, No. 3, 259-283 (1978).
15. F. M. Wiegel, "A network model for viscoelastic fluids," Physica, 42, No. 1, 156-164 (1969).
16. A. Kaye, "An equation of state for non-Newtonian fluids," Brit. J. Appl. Phys., 17, No. 6, 803-806 (1966).
17. M. Takahashi, T. Masuda, and S. Onogi, "New critical tests for integral constitutive equations," Trans. Soc. Rheology, 21, No. 3, 337-354 (1977).
18. M. Takahashi, T. Masuda, and S. Onogi, "A stress dependent constitutive equation," J. Rheology, 22, 285-291 (1978).
19. R. B. Bird, O. Hassager, R. C. Armstrong, and C. C. Curtiss, Dynamics of Polymeric Fluids, Vol. II, Wiley, New York (1977).
20. R. I. Tanner, "Stresses in dilute solutions of bead-nonlinear-spring macromolecules, III. Friction coefficient varying with dumbell extension," Trans. Soc. Rheology, 19, No. 4, 557-582 (1975).
21. N. Phan-Thien and R. I. Tanner, "Langevin analysis of dumbbells with hydrodynamic interaction," Rheol. Acta, 17, No. 6, 568-577 (1978).
22. E. J. Hinch, "Mechanical models of dilute polymer solutions in strong flows," Phys. Fluids, 20, No. 10, Pt. II, 22-30 (1977).
23. R. C. Armstrong and S. Ishikawa, "A rheological equation of state for dilute solutions of nearly Hookean dumbells," J. Rheology, 24, 143-165 (1980).
24. A. I. Leonov, "Nonequilibrium thermodynamics and rheology of viscoelastic polymer media," Rheol. Acta, 15, No. 2, 85-98 (1976).
25. C. Pearson and S. Middleman, "An empirical constitutive model for polymer solutions," Trans. Soc. Rheol., 20, No. 4, 559-573 (1976).
26. J. David, "On some integral viscoelastic fluid models with the rate-of-strain dependent function," Rheol. Acta, 11, No. 3/4, 333-340 (1972).
27. C. J. S. Petrie, "Measure of deformation and convected derivatives," J. Non-Newtonian Fluid Mech., 5, 147-176 (1979).
28. J. D. Goddard, "Polymeric fluid mechanics," Adv. Appl. Mech., 19, 143-219 (1979).